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A Note on Strongly $(\Delta_{(r)})^{\lambda}$ - Summable And $(\Delta_{(r)})^{\lambda}$ - Statistical Convergence Sequences Of Fuzzy Numbers

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A Note on Strongly $(\Delta_{(r)})^{\lambda}$ - Summable And $(\Delta_{(r)})^{\lambda}$ - Statistical Convergence Sequences Of Fuzzy Numbers

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I. Introduction

The idea of the statistical convergence of sequence was introduced by Fast [4] and Schoenberg [12] independently in order to extend the notion of convergence of sequences. It is also found in Zygmund [16]. Later on it was linked with summability by Fridy and Orhan [5], Maddox [9] and many others. In [11] Nuray and Savaş extended the idea to sequences of fuzzy numbers and discussed the concept of statistically Cauchy sequences of fuzzy numbers. On strongly λ-summability and λ-statistical convergence can be found in [14]. In this article we extend these notions to difference sequences of fuzzy numbers.

Let $C(R^n) = \{A \subset R^n$: A compact and convex$\}$. The space $C(R^n)$ has a liner structure induced by the operations $A+B = \{a + b: a \in A, b \in B\}$ and $\lambda A = \{\lambda a: a \in A\}$ for $A, B \in C(R^n)$ and $\lambda \in R$. The Hausdroff distance between A and B of $C(R^n)$ is defined as:

$$\delta_{\infty}(A,B) = \max\left\{\sup_{a\in A}\inf_{b\in B}\|a-b\|, \sup_{b\in B}\inf_{a\in A}\|a-b\|\right\}.$$

Let $L(R^n)$ denote the set of all fuzzy numbers. The linear structure of $L(R^n)$ induces addition $X + Y$ and scalar multiplication λX, $\lambda \in R$, in terms of α-level sets, by

$[X+Y]^{\alpha} = [X]^{\alpha} + [Y]^{\alpha}$ and $[\lambda X]^{\alpha} = \lambda [X]^{\alpha}$ for each $0 \le \alpha \le 1$, where the α-level set $[X]^{\alpha} = \left\{x \in R^n : X(x) \ge \alpha\right\}$ is a nonempty compact and convex subset of R^n and X is a fuzzy number i.e., a function from R^n to [0, 1] which is normal, fuzzy convex, upper semi-continuous and the closure $X^0 = \left\{x \in R^n : X(x) > 0\right\}$ is compact.

Define for each $1 \le q < \infty$

$$d_q(X,Y) = \left(\int_0^1 \delta_{\infty}(X^{\alpha}, Y^{\alpha})^q d_{\infty}\right)^{1/q}$$

And $d_{\infty} = \sup_{0\le\alpha\le1} \delta_{\infty}(X^{\alpha}, Y^{\alpha})$. Clearly $d_{\infty}(X,Y) = \lim_{q\to\infty} d_q(X,Y)$ with $d_q \le d_r$ if $q \le r$. Moreover d_q is a complete, separable and locally compact metric space (see [1]).

Throughout the paper, d will denote d_q with $1 \le q < \infty$.

We now state the following definitions which can be found in [8, 11, 13].

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Global Journal of Science Frontier Research (F) Volume XII Issue I Version I January 2012

A sequence $X = (X_k)$ of fuzzy numbers is a function X from the set N of all positive integers into $L(R)$. The fuzzy number X_k denotes the value of the function at $k \in N$ and is called the k-th term or general term of the sequence.

A sequence $X = (X_k)$ of fuzzy numbers is said to be convergent to the fuzzy number X_0, written as $\lim_k X_k = X_0$, if for every $\varepsilon > 0$ there exists $n_0 \in N$ such that

$$d(X_k, X_0) < \varepsilon \text{ for } k > n_0$$

Again $X = (X_k)$ is said to be a Cauchy sequence if for every $\varepsilon > 0$ there exists $n_0 \in N$ such that

$$d(X_k, X_l) < \varepsilon \text{ for } k, l > n_0$$

A sequence $X = (X_k)$ of fuzzy numbers is said to be bounded if the set $\{X_k : k \in N\}$ of fuzzy numbers is bounded.

The natural density of a set K of positive integers is denoted by $\delta(K)$ and defined by

$$\delta(K) = \lim_n \frac{1}{n} \text{card}\{k \le n : k \in K\}$$

A sequence $X = (X_k)$ of fuzzy numbers is said to be statistically convergent to a fuzzy number X_0 if for every $\varepsilon > 0$, $\lim_n \frac{1}{n} \text{card}\{k \le n : d(X_k, X_0) \ge \varepsilon\} = 0$ and we write st-lim $X_k = X_0$.

Let Z be a real sequence space, then Kizmaz [7] introduced the following difference sequence spaces:

$$Z(\Delta) = \{(x_k) \in w : (\Delta x_k) \in Z\},$$

for $Z = \ell_\infty, c, c_0$, where $\Delta x_k = x_k - x_{k+1}$, for all $k \in N$.

II. New Definitions and Main Results

In this section we define some new definitions and investigate the main results of this article.

Let r be a non-negative integer. Let $\lambda = (\lambda_k)$ be a non-decreasing sequence of positive numbers tending to ∞ and $\lambda_{n+1} \le \lambda_n + 1$, $\lambda_1 = 1$. Then the sequence $X = (X_k)$ of fuzzy numbers is said to be strongly $\left(\Delta_{(r)}\right)^{\lambda}$ - summable to a fuzzy number X_0 if

$\lim_{n\to\infty} \frac{1}{\lambda_n} \sum_{k \in I_n} d\left(\Delta_{(r)} X_k, X_0\right) = 0$, where $I_n = [n - \lambda_n + 1, n]$ and $\left(\Delta_{(r)} X_k\right) = \left(X_k - X_{k-r}\right)$ and $\Delta_{(0)} X_k = X_k$ for all $k \in N$. For details about the operator, one can refer to Dutta [2, 3]

In this expansion it is important to note that we take $X_k = \bar{0}$ for non-positive values of k.

If we take $r = 0$, then strongly $\left(\Delta_{(r)}\right)^{\lambda}$- summability reduces to strongly λ- summability. It is clear that strongly λ-summability implies strongly $\left(\Delta_{(r)}\right)^{\lambda}$-summability.

In particular if we take $\lambda_n = n$, for all $n \in N$ then we say $X = (X_k)$ is strongly $\Delta_{(r)}$ - Cesàro summable to X_0.

A sequence $X = (X_k)$ of fuzzy numbers is said to be $\left(\Delta_{(r)}\right)^{\lambda}$ - statistically convergent to a fuzzy number X_0 if for every $\varepsilon > 0$

$$\lim_n \frac{1}{\lambda_n} \text{card}\{k \in I_n : d(\Delta_{(r)} X_k, X_0) \ge \varepsilon\} = 0$$

In particular if we take $\lambda_n = n$, for all $n \in N$, then we say that $X = (X_k)$ is $\Delta_{(r)}$ - statistically convergent to X_0.

Global Journal of Science Frontier Research (F) Volume XII Issue I Version I

January 2012

Again if we take $\lambda_n = n$, for all $n \in N$, $r = 0$, then $\left(\Delta_{(r)}\right)^{\lambda}$ - statistically convergence reduces to statistically convergence. Our next aim is to present some relationship between strongly $\left(\Delta_{(r)}\right)^{\lambda}$ - summability and $\left(\Delta_{(r)}\right)^{\lambda}$ - statistically convergent.

Theorem 2.1. *If a sequence $X = (X_k)$ is strongly $\left(\Delta_{(r)}\right)^{\lambda}$ - summable then it is $\left(\Delta_{(r)}\right)^{\lambda}$ - statistically convergent.*

Proof. Suppose $X = (X_k)$ is strongly $\left(\Delta_{(r)}\right)^{\lambda}$ - summable to X_0. Then

$$\lim_{n\to\infty} \frac{1}{\lambda_n} \sum_{k\in I_n} d\left(\Delta_{(r)} X_k, X_0\right) = 0.$$

Now the result follows from the following inequality:

$$\sum_{k\in I_n} d\left(\Delta_{(r)} X_k, X_0\right) \geq \varepsilon \operatorname{card}\left\{k \in I_n : d(\Delta_{(r)} X_k, X_0) \geq \varepsilon\right\}$$

Theorem 2.2. *If a sequence $X = (X_k)$ is $\Delta_{(r)}$ - bounded and $\left(\Delta_{(r)}\right)^{\lambda}$ - statistically convergent then it is strongly $\left(\Delta_{(r)}\right)^{\lambda}$ - summable.*

Proof. Suppose $X = (X_k)$ is $\Delta_{(r)}$ -bounded and $\left(\Delta_{(r)}\right)^{\lambda}$ - statistically convergent to X_0. Since $X = (X_k)$ is $\Delta_{(r)}$ - bounded, we can find a fuzzy number M such that $d(\Delta_{(r)} X_k, X_0) \leq M$ for all $k \in N$

Again since $X = (X_k)$ is $\left(\Delta_{(r)}\right)^{\lambda}$ - statistically convergent to X_0, for every $\varepsilon > 0$

$$\lim_{n} \frac{1}{\lambda_n} \operatorname{card}\left\{k \in I_n : d(\Delta_{(r)} X_k, X_0) \geq \varepsilon\right\} = 0$$

Now the result follows from the following inequality:

$$\frac{1}{\lambda_n} \sum_{k\in I_n} d\left(\Delta_{(r)} X_k, X_0\right) = \frac{1}{\lambda_n} \sum_{\substack{k\in I_n \\ d(\Delta_{(r)} X_k, X_0) \geq \varepsilon}} d(\Delta_{(r)} X_k, X_0) + \frac{1}{\lambda_n} \sum_{\substack{k\in I_n \\ d(\Delta_{(r)} X_k, X_0) < \varepsilon}} d(\Delta_{(r)} X_k, X_0)$$

$$\leq \frac{M}{\lambda_n} \operatorname{card}\left\{k \in I_n : d(\Delta_{(r)} X_k, X_0) \geq \varepsilon\right\} + \varepsilon$$

Corollary 2.3. *If a sequence $X = (X_k)$ is $\Delta_{(r)}$ - bounded and $\left(\Delta_{(r)}\right)^{\lambda}$ - statistically convergent then it is strongly $\Delta_{(r)}$ - Cesàro summable.*

Proof. Proof follows by combining the above Theorem and the following inequality:

$$\frac{1}{n} \sum_{k=1}^{n} d\left(\Delta_{(r)} X_k, X_0\right) = \frac{1}{n} \sum_{k=1}^{n-\lambda_n} d\left(\Delta_{(r)} X_k, X_0\right) + \frac{1}{n} \sum_{k\in I_n} d\left(\Delta_{(r)} X_k, X_0\right)$$

$$\leq \frac{1}{\lambda_n} \sum_{k=1}^{n-\lambda_n} d\left(\Delta_{(r)} X_k, X_0\right) + \frac{1}{\lambda_n} \sum_{k\in I_n} d\left(\Delta_{(r)} X_k, X_0\right)$$

$$\leq \frac{2}{\lambda_n} \sum_{k\in I_n} d\left(\Delta_{(r)} X_k, X_0\right)$$

Theorem 2.4. *If a sequence $X = (X_k)$ is $\Delta_{(r)}$ - statistically convergent and* $\liminf_n \left(\frac{\lambda_n}{n}\right) > 0$ *then it is $\left(\Delta_{(r)}\right)^{\lambda}$ - statistically convergent.*

Global Journal of Science Frontier Research (F) Volume XII Issue I Version I January 2012

Proof. Assume the given conditions. For a given $\varepsilon > 0$, we have

$$\left\{k \in I_n : d(\Delta_{(r)} X_k, X_0) \geq \varepsilon\right\} \subset \left\{k \leq n : d(\Delta_{(r)} X_k, X_0) \geq \varepsilon\right\}$$

Hence the proof follows from the following inequality:

$$\frac{1}{n}\operatorname{card}\left\{k \leq n : d(\Delta_{(r)} X_k, X_0) \geq \varepsilon\right\} \geq \frac{1}{n}\operatorname{card}\left\{k \in I_n : d(\Delta_{(r)} X_k, X_0) \geq \varepsilon\right\}$$

$$= \frac{\lambda_n}{n}\frac{1}{\lambda_n}\operatorname{card}\left\{k \in I_n : d(\Delta_{(r)} X_k, X_0) \geq \varepsilon\right\}$$

Remark. It is easy to see that if a sequence $X = (X_k)$ is bounded then it is $\Delta_{(r)}$ - bounded. If $X = (X_k)$ is λ-statistically convergent then it is $\left(\Delta_{(r)}\right)^{\lambda}$-statistically convergent. Again if $X = (X_k)$ is strongly λ- summable then it is strongly $\left(\Delta_{(r)}\right)^{\lambda}$ - summable. Therefore we can replace the phrases 'if a sequence $X = (X_k)$ is strongly $\left(\Delta_{(r)}\right)^{\lambda}$ - summable' by 'if a sequence $X = (X_k)$ is strongly λ- summable', 'if a sequence $X = (X_k)$ is $\Delta_{(r)}$ - bounded and $\left(\Delta_{(r)}\right)^{\lambda}$ - statistically convergent' by 'if a sequence $X = (X_k)$ is bounded and λ - statistically convergent', 'if a sequence $X = (X_k)$ is $\Delta_{(r)}$ - bounded and $\left(\Delta_{(r)}\right)^{\lambda}$ - statistically convergent' by 'if a sequence $X = (X_k)$ is bounded and λ - statistically convergent' and 'if a sequence $X = (X_k)$ is $\Delta_{(r)}$ - statistically convergent' by 'if a sequence $X = (X_k)$ is statistically convergent' respectively in Theorem 2.1, Theorem 2.2, Corollary 2.3 and Theorem 2.4.

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